Computers and Mathematics with Applications 60 (2010) 1868–1872

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A modified variational iteration method for solving Riccati differential equations

Fazhan Geng

Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, China

ARTICLE INFO

Article history:
Received 3 May 2010
Received in revised form 13 July 2010
Accepted 13 July 2010

Keywords:
Modified variational iteration method
Riccati differential equation

ABSTRACT

In this paper, we introduce a modified variational iteration method (MVIM) for solving Riccati differential equations. The solutions of Riccati differential equations obtained using the traditional variational iteration method (VIM) give good approximations only in the neighborhood of the initial position. The main advantage of the present method is that it can enlarge the convergence region of iterative approximate solutions. Hence, the solutions obtained using the MVIM give good approximations for a larger interval, rather than a local vicinity of the initial position. Numerical results show that the method is simple and effective.

1. Introduction

In this paper, a modified variational iteration method is presented for addressing the following Riccati differential equation:

$$\begin{cases} u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), & 0 \le x \le X, \\ u(0) = \alpha, \end{cases} \tag{1.1}$$

where $p(x), q(x), r(x)$ are continuous, which plays a significant role in many fields of applied science [1]. For example, as is well-known, a one-dimensional static Schrödinger equation is closely related to a Riccati differential equation. A solitary wave solution of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [2]. Such problems also arise in the optimal control literature. Therefore, the problem has attracted much attention and has been studied by many authors. However, an analytical solution in an explicit form seems unlikely to be found except for certain special situations. For example, some Riccati equations with constant coefficients can be solved analytically by various methods [3]. Therefore, one has to resort to numerical techniques or approximate approaches to get its solution. Recently, Adomian's decomposition method has been proposed for solving Riccati differential equations in [4]. Abbasbandy [5–7] solved a special Riccati differential equation—the quadratic Riccati differential equation—using He's VIM, the homotopy perturbation method (HPM) and the iterated He's HPM, and compared the accuracy of the solution obtained with that derived by Adomian's decomposition method. Geng [8] introduced the piecewise VIM for solving Riccati differential equations.

The variational iteration method, which was proposed originally by He [9–14], has been proved by many authors to be a powerful mathematical tool for addressing various kinds of linear and nonlinear problems [15–25]. The reliability of the method and the reduction in the burden of computational work gave this method wider application.

In this paper, we present a MVIM for solving (1.1) and obtain an accurate numerical solution. The advantage of the MVIM over the existing methods for solving this problem is that the solution of (1.1) obtained using the present method is efficient not only for a smaller value of x but also for a larger value.

E-mail address: gengfazhan@sina.com.

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.07.017

The rest of the paper is organized as follows. In the next section, the VIM is introduced. The MVIM for solving (1.1) is presented in Section 3. Numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Analysis of the variational iteration method

Consider the differential equation

$$Lu + Nu = g(x), \tag{2.1}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source inhomogeneous term. In [9–14], the VIM was introduced by He, where a correct functional for (2.1) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(t) + N\widetilde{u}_n(t) - g(t)\}\mathrm{d}t, \tag{2.2}$$

where λ is a general Lagrangian multiplier [10], which can be identified optimally via variational theory, and $\widetilde{u}_n$ is a restricted variation which means that $\delta\widetilde{u}_n = 0$. By this method, it is firstly required to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximations $u_{n+1}(x),\ n \geq 0$, of the solution $u(x)$ will be readily obtained upon using the Lagrangian multiplier determined and any selective function $u_0(x)$. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x).$$

In fact, the solution of problem (2.1) is considered as a fixed point of the following functional under a suitable choice of the initial term $u_0(x)$:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(t) + Nu_n(t) - g(t)\}\mathrm{d}t. \tag{2.3}$$

As a well-known powerful tool, we have:

Theorem 2.1 (*Banach's Fixed Point Theorem*)**.** *Assume that X is a Banach space and*

$$A : X \to X$$

is a nonlinear mapping, and suppose that

$$\|A[u] - A[v]\| \leq \alpha\|u - v\|, \quad u, v \in X$$

for some constants $\alpha < 1$. Then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A[u_n],$$

with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A.

According to Theorem 2.1, for the nonlinear mapping

$$A[u(x)] = u(x) + \int_0^x \lambda\{Lu(t) + Nu(t) - g(t)\}\mathrm{d}t,$$

a sufficient condition for convergence of the variational iteration method is strict contraction of A. Furthermore, the sequence (2.3) converges to the fixed point of A which is also the solution of problem (2.1).

3. The modified variational iteration method for solving (1.1)

The main drawback of the standard VIM is that the sequence of successive approximations of the solution obtained can be rapidly convergent only in a small region, which will greatly restrict the application area of such a method.

To enlarge the convergence region of the sequence of successive approximations obtained, we shall modify the VIM by introducing an auxiliary parameter.

For (2.1), we rewrite it as

$$Lu - Lu + \gamma[Lu + Nu - g(x)] = 0, \tag{3.1}$$

where γ is an auxiliary parameter and $\gamma \neq 0$, which is used to adjust the convergence region of the following iterative formula.

A correct functional for (2.1) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(t) - L\widetilde{u}_n(t) + \gamma[L\widetilde{u}_n(t) + N\widetilde{u}_n(t) - g(t)]\}\mathrm{d}t, \tag{3.2}$$

where λ is a general Lagrangian multiplier, which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means that $\delta\tilde{u}_n = 0$.

According to the VIM, the following iteration formula can be obtained:

$$u_{n+1}(x) = u_n(x) + \gamma \int_0^x \lambda(t,x)[Lu_n(t) + Nu_n(t) - g(t)]\mathrm{d}t, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

From the convergence analysis in Section 2, it is easy to see that the smaller the value of $|\gamma|$ is, the wider the convergence region of iterative sequence (3.3) is.

In fact, iterative formula (3.3) gives us vast freedom of choice. For some strong nonlinear problems, one can choose a relatively small value of $|\gamma|$ (generally less than 1) to obtain a good approximation in a wider region.

In addition, it should be especially pointed out that when $\gamma = 1$, (3.3) becomes the standard variational iteration formula (2.3).

For Eq. (1.1), according to the above MVIM, we construct the correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\{u_n'(t) - \tilde{u}_n'(t) + \gamma[\tilde{u}_n'(t) - q(t)\tilde{u}_n(t) - r(t)\tilde{u}_n^2(t) - p(t)]\}\mathrm{d}t, \quad 0 \le x \le X, \tag{3.4}$$

where $\tilde{u}_n$ is a restricted variation, i.e. $\delta\tilde{u}_n = 0$; λ is a general Lagrangian multiplier and can be easily identified as $\lambda = -1$.

So we can obtain the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \gamma \int_0^x [u_n'(t) - q(t)u_n(t) - r(t)u_n^2(t) - p(t)]\mathrm{d}t, \quad 0 \le x \le X,\ n = 0, 1, 2, \ldots \tag{3.5}$$

where $u_0(x)$ is an initial approximation satisfying the initial condition of Eq. (1.1).

Theorem 3.1. *Suppose that $u_0(x) = \alpha$ and the iterative sequence $\{u_n(x)\}$ obtained from (3.5) converge to $u(x)$; then $u(x)$ is the solution of Eq. (1.1).*

Proof. Taking limits in the iterative formula in (3.5), it follows that

$$\lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty} u_n(x) - \gamma \int_0^x \lim_{n\to\infty} [u_n'(t) - q(t)u_n(t) - r(t)u_n^2(t) - p(t)]\mathrm{d}t, \tag{3.6}$$

and thus,

$$\gamma \int_0^x [u'(t) - q(t)u(t) - r(t)u^2(t) - p(t)]\mathrm{d}t = 0. \tag{3.7}$$

Since $\gamma \neq 0$, it follows immediately that

$$\int_0^x [u'(t) - q(t)u(t) - r(t)u^2(t) - p(t)]\mathrm{d}t = 0. \tag{3.8}$$

Then differentiation of both sides with respect to x yields

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x). \tag{3.9}$$

Obviously, $u(x)$ satisfies Eq. (1.1). Also, $u(0) = \alpha$, since $u_n(0) = \alpha$.

Hence, $u(x)$ is the solution of Eq. (1.1) and the proof is complete. □

According to Banach's fixed point theorem, it is easy to obtain the convergence condition for the sequence u_n obtained from (3.5).

Theorem 3.2. *Define a nonlinear mapping*

$$T[u(x)] = u(x) - \gamma \int_0^x [u'(t) - q(t)u(t) - r(t)u^2(t) - p(t)]\mathrm{d}t.$$

A sufficient condition for convergence of the iterative sequence $\{u_n(x)\}$ obtained from (3.5) is strict contraction of the nonlinear mapping T. Furthermore, the sequence (3.5) converges to the fixed point of T which is also the solution of Eq. (1.1).

Therefore, according to (3.5), by choosing a proper γ and initial approximation $u_0(x)$, the successive approximations of the solution to (1.1) on the entire interval $[0, X]$ can be obtained.

4. Numerical examples

Now we apply the MVIM presented in Section 3 to some Riccati differential equations. Numerical results show that the MVIM is very effective.

Table 1
Numerical results for Example 4.1.

x	Exact solution $u(x)$	Present method $(u_9(x))$	VIM $(u_2(x))$	VIM $(u_9(x))$
0.4	0.567812	0.513543	0.538667	0.567812
1.2	1.95136	1.90195	2.064	1.95136
2.0	2.35777	2.41229	3.33333	-1.34×10^{13}
2.8	2.40823	2.30603	3.32267	-1.10×10^{59}
3.6	2.41359	2.40026	1.008	-3.10×10^{100}
4.0	2.41401	2.50735	−1.33333	-4.90×10^{119}

Table 2
Numerical results for Example 4.2.

x	Exact solution $u(x)$	Present method $(u_{10}(x))$	Method in [8]	VIM $(u_2(x))$	VIM $(u_{10}(x))$
0.4	1.01765	1.0153	1.07252	1.01704	1.01765
0.8	1.11809	1.10893	1.14806	1.09907	1.11809
1.2	1.33114	1.32233	1.38805	1.17352	1.33113
2.0	2.00973	2.04175	2.10337	−1.03075	3.45×10^{9}
2.8	2.80021	2.76833	2.90001	−23.3443	-1.88×10^{164}
3.6	3.60000	3.56075	3.7	−135.829	0
4.0	4.00000	4.09113	4.1	−280.397	0

Example 4.1. Consider the following Riccati differential equation [4–8]:

$$\begin{cases} u'(x) = 1 + 2u(x) - u^2(x), \quad 0 \le x \le 4, \\ u(0) = 0. \end{cases} \tag{4.1}$$

The exact solution can be easily determined to be

$$u(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{\log\left(\frac{-1+\sqrt{2}}{1+\sqrt{2}}\right)}{2}\right).$$

According to (3.5), taking $\gamma = 0.3,\ n = 9$, the numerical results are shown in Table 1. From Table 1, we find that the solution derived by the VIM [6] gives a good approximation only in the neighborhood of the initial position, while the present method gives a good approximation in a wider region.

Remark. The solutions of Example 4.1 derived using the ADM [4], HPM [5] and VIM [6] give good approximations only in the neighborhood of the initial position. The approximations derived by the present MVIM, iterated HPM [7] and piecewise VIM [8] are all efficient for the whole interval. However, the present method is more accurate than iterated HPM [7].

Example 4.2. Consider the following Riccati differential equation [8]:

$$\begin{cases} u'(x) = 1 + x^2 - u^2(x), \quad 0 \le x \le 4, \\ u(0) = 1, \end{cases} \tag{4.2}$$

with the exact solution

$$u(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2}\,dt}.$$

According to (3.5), taking $\gamma = 0.15,\ n = 10$, we can obtain the approximations of (4.2) on [0, 4]. The numerical results are shown in Table 2. For analyzing the influence of the value of parameter γ, taking $n = 5$, the comparison of relative errors for different values of the parameter γ is shown in Table 3. From Table 3, it is easy to see that the smaller the value of $|\gamma|$ is, the wider the convergence region of the iterative sequence (3.5) is. Also, when the value of $|\gamma|$ is low, the convergence rate of the iterative formula is relatively slow, and so more iterative steps are required.

5. Conclusion

In this paper, a MVIM is presented for solving Riccati differential equations. Comparing with the standard VIM results, the results for numerical examples demonstrate that the present method can give a more accurate approximation in a larger region. This is also the main advantage of the present method. Therefore, the modification of the VIM can overcome the restriction of the application area of the VIM, and then expand its scope of application. However, generally, when the value of $|\gamma|$ chosen is small, the rate of convergence of the iterative formula is relatively slow, and so more iterative steps are required. This is the drawback of this modification.

Table 3
Comparison of relative errors for different value of γ for Example 4.2.

x	$u(x)$	Relative errors ($\gamma = 0.15$)	Relative errors ($\gamma = 0.3$)	Relative errors ($\gamma = 0.5$)	Relative errors ($\gamma = 1.0$)
0.4	1.01765	0.006357	0.001782	0.000137	9.7×10^{-7}
0.8	1.11809	0.030422	0.005078	0.000275	0.000252
1.2	1.33114	0.049773	0.000408	0.000724	0.007552
2.0	2.00973	0.002050	0.012415	0.000246	0.588025
2.8	2.80021	0.072808	0.036023	0.011100	3388.290
3.6	3.60000	0.015616	0.055719	0.365917	8.2×10^{10}
4.0	4.00000	0.098313	0.140001	496.7350	5.5×10^{13}

Acknowledgements

The authors would like to express thanks to the unknown referees for their careful reading and helpful comments.

References

[1] W.T. Reid, Riccati Differential Equations, Academic Press, New York, 1972.
[2] J.F. Carinena, G. Marmo, A.M. Perelomov, M.F. Ranada, Related operators and exact solutions of Schrodinger equations, International Journal of Modern Physics A 13 (1998) 4913–4929.
[3] M.R. Scott, Invariant Imbedding and its Applications to Ordinary Differential Equations, Addison-Wesley, 1973.
[4] M.A. El-Tawil, A.A. Bahnasawi, A. Abdel-Naby, Solving Riccati differential equation using Adomian's decomposition method, Applied Mathematics and Computation 157 (2004) 503–514.
[5] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method, Applied Mathematics and Computation 172 (2006) 485–490.
[6] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, Journal of Computational and Applied Mathematics 207 (2007) 59–63.
[7] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, Applied Mathematics and Computation 175 (2006) 581–589.
[8] F.Z. Geng, Y.Z. Lin, M.G. Cui, A piecewise variational iteration method for Riccati differential equations, Computers and Mathematics with Applications 58 (2009) 2518–2522.
[9] J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics 34 (1999) 699–708.
[10] J.H. He, Variational iteration method for autonomous ordinary differential system, Applied Mathematics and Computation 114 (2000) 115–123.
[11] J.H. He, Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics B 20 (2006) 1141–1199.
[12] J.H. He, X.H. Wu, Variational iteration method: new development and applications, Computers and Mathematics with Applications 54 (2007) 881–894.
[13] J.H. He, Variational iteration method—some recent results and new interpretations, Journal of Computational and Applied Mathematics 207 (2007) 3–17.
[14] J.H. He, G.C. Wu, F. Austin, The variational iteration method which should be followed, Nonlinear Science Letters A 1 (1) (2010) 1–30.
[15] J.F. Lu, Variational iteration method for solving two-point boundary value problems, Journal of Computational and Applied Mathematics 207 (2007) 92–95.
[16] A.A. Soliman, M.A. Abdou, Numerical solutions of nonlinear evolution equations using variational iteration method, Journal of Computational and Applied Mathematics 207 (2007) 111–120.
[17] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burgers and coupled Burgers equations, Journal of Computational and Applied Mathematics 181 (2005) 245–251.
[18] D.D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction–diffusion equations, International Journal of Nonlinear Sciences and Numerical Simulation 7 (4) (2006) 411–418.
[19] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, International Journal of Nonlinear Sciences and Numerical Simulation 7 (2006) 27–34.
[20] D.H. Shou, J.H. He, Application of parameter-expanding method to strongly nonlinear oscillators, International Journal of Nonlinear Sciences and Numerical Simulation 8 (1) (2007) 121–124.
[21] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 7 (1) (2006) 65–70.
[22] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, Journal of Computational and Applied Mathematics 207 (2007) 121–128.
[23] J. Biazar, H. Ghazvini, He's variational iteration method for solving hyperbolic differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (3) (2007) 311–314.
[24] H. Ozer, Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics, International Journal of Nonlinear Sciences and Numerical Simulation 8 (4) (2007) 513–518.
[25] E. Yusufoglu, Variational iteration method for construction of some compact and noncompact structures of Klein–Gordon equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 153–158.